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Computational aspects of affine representations for torsion free nilpotent groups via the Seifert construction

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Abstract

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We study and describe faithful affine representations of a certain canonical form for torsion free, finitely generated nilpotent groups. The algebraic set-up used is that of the Seifert Fiber Space Construction conceived by Kyung Bai Lee and Frank Raymond, and can be considered as suitable for an iteration procedure. The representations obtained realise such a group as the fundamental group of a compact, complete affinely flat manifold. We investigate computational aspects for explicit constructions of these representations. An equivalent description for canonical form representations, in terms of matrices over polynomial rings, is presented. As these groups are uniform lattices in nilpotent Lie groups, it is interesting to draw the related picture on the Lie algebra level. An example (due to Dan Segal and Fritz Grunewald) is used to illustrate that the iteration aspect of this set-up should be well understood and will need a very careful treatment. Finally, we present canonical representations for several infinite families of groups, thus also giving positive evidence for the nilpotent case of a conjecture of Milnor's.

1. Introduction

In this paper torsion free, finitely generated nilpotent groups N will be our main point of interest. It is well known that these groups occur as the uniform lattices in simply connected, connected nilpotent Lie groups.

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In 1983, Lee [4] proved that every torsion free, finitely generated virtually 3-step nilpotent group is the fundamental group of a compact, complete affinely flat manifold. The main idea in his approach was to consider these groups as extensions which could be treated as a particular case of what is called the Seifert Fiber Space Construction. As a result it followed that those groups could be embedded faithfully in the group of affine motions of Euclidean m-space ($m = \mathrm{rank}(N)$), acting properly discontinuously and with compact quotient.

The spirit behind this construction was the (tempting) idea that, if there is an embedding of such a group N into $\mathrm{Aff}(\mathbb{R}^m) = \mathbb{R}^m \rtimes \mathrm{GL}(m, \mathbb{R})$ (where $m = \mathrm{rank}(N)$), the center of N might be mapped into the group of pure translations. Based on this idea, it becomes natural to try an iteration process, building up affine representations step by step, by central extensions, from the abelian case towards higher nilpotency class. Using the construction of Lee, the affine representations for N are of a nice, quite well-understood type, which has been called 'special' in [3], or more recently 'canonical' [7].

Lee's result provided the first systematic evidence for a conjecture of Milnor [6], stating that every torsion free virtually polycyclic group is the fundamental group of a compact complete affinely flat manifold M. In this text we shall consider an iteration of the Seifert fiber space construction based on central extensions, as conceived by Lee. However, our point of view will be a computational one. We present an equivalent description of canonical type representations in terms of unitriangular matrices with polynomial entries. A few infinite families of groups and their representations will be given.

In Section 4 we draw the link with the corresponding Lie algebra representations which are also of a nice 'canonical' type. Here, we note that in 1989, Boyom [1] published a positive answer to the nilpotent case of Milnor's conjecture by claiming—also in an iterative set up—the existence of certain Lie algebra representations (left symmetric or Koszul–Vinberg (KV) structures). However, by means of an example (communicated to us by D. Segal and F. Grunewald), we will show that his main theorem, 'The Lifting Theorem', is stated incorrectly. The same example—on the group level now—will be used to indicate that the iteration idea mentioned above will not work in complete generality and needs a very careful treatment.

2. Preliminaries

Let us start with some notational remarks. For any group A, $Z(A)$ will denote the center of A. In the sequel, we will work frequently with commutators in nilpotent groups. We will write $[a, b]$ for the commutator of a and b, which is meant to be $a^{-1}b^{-1}ab$, such that $ab = ba[a, b]$.

We use the real vector space of affine transformations $\mathbb{R}^n \to \mathbb{R}^k$ which we denote by $\mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^k)$. We write $\mathrm{Aff}(\mathbb{R}^n)$ for the subgroup (under composition) of $\mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^n)$ consisting of invertible affine mappings. Affine transformations λ ($\in \mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^k)$) will be written as consisting of a linear ($(k \times n)$-matrix) part D ($=\mathrm{Lin}(\lambda)$) and of a translational part d ($=\mathrm{Tr}(\lambda)$), together denoted by $\lambda = (D, d)$.

Whenever we write N in the sequel, we will mean a finitely generated, torsion free nilpotent group of class c. Let $N = N_1$ and $N_{i+1} = N_i/Z(N_i)$ ($i \geq 1$). Then $N_{c+1} = N_c/Z(N_c) = 1$. Let $\mathrm{rank}(Z(N_i)) = k_i$ and write $K_i = \sum_{j \geq i} k_j$. We also write K for K_1, which is the rank or Hirsch number of N. N as given here, will be called a group of type $(k_1, k_2, \ldots, k_c)$.

It is a well-known fact that N can be given a commutator presentation,

$$\langle a_1, \ldots, a_K \mid [a_i, a_j] = \text{word in generators } a_k,\, k > i,\, i > j \rangle\,. \tag{1}$$

In such a presentation, as usual, we write down only the nontrivial commutators. All commutators not explicitly written, are supposed to be trivial.

Each element $n \in N$ can now be written uniquely as a word $n = a_1^{x_1} \ldots a_K^{x_K}$ called the normal form of n; as a consequence we will often identify n with a unique coordinate vector $(x_1, \ldots, x_K) \in \mathbb{Z}^K$.

Definition 2.1. A faithful representation $\rho : N \to \mathrm{Aff}(\mathbb{R}^K)$ will be called '*in canonical form*' (or a *canonical representation*) if and only if

(1) the matrix parts of ρ are blocked upper-triangular with the identity matrices of size $k_1, k_2, \ldots, k_c$ as diagonal entries,

(2) the subgroup of N corresponding to $Z(N_i)$ acts on the ith block ($\mathbb{R}^{k_i}$) as translations ($\cong \mathbb{Z}^{k_i}$ and with compact quotient) and trivially on $\mathbb{R}^{K_{i+1}}$.

3. On iterating canonical representations via the Seifert construction

Consider a group N with a canonical representation $\rho : N \to \mathrm{Aff}(\mathbb{R}^K)$, $n \mapsto \rho(n) = (A(n), a(n))$. Regard $\mathbb{Z}^k$ as a subgroup of constant mappings in $\mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)$, e.g., z ($\in \mathbb{Z}^k$) $\mapsto (0, z)$ ($\in \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)$).

There is an action of $\mathrm{Aff}(\mathbb{R}^K)$ (and so of N) on $\mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)$ as follows: if $h \in \mathrm{Aff}(\mathbb{R}^K)$ and $\lambda \in \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)$ then ${}^h\lambda = \lambda \circ h^{-1}$. Clearly $\mathbb{Z}^k$ becomes a trivial N-module. The semi-direct product $\mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k) \rtimes \mathrm{Aff}(\mathbb{R}^K)$ acts on $\mathbb{R}^k \times \mathbb{R}^K = \mathbb{R}^{k+K}$: we define for $(x, y) \in \mathbb{R}^k \times \mathbb{R}^K$, ${}^{(\lambda, h)}(x, y) = (x + \lambda(h(y)), h(y))$. It follows immediately that if $h = (A, a)$ and $\lambda = (D, d)$, this action is given by

$$\begin{pmatrix} I & DA \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} Da + d \\ a \end{pmatrix} \tag{2}$$

and so is clearly affine.

The iteration problem ('universal' setting). Given the previous set-up, it is natural to consider the following problem: assume a central extension $1 \to \mathbb{Z}^k \to$

$E \to N \to 1$, can we extend ρ to a representation $\rho' : E \to \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k) \rtimes \mathrm{Aff}(\mathbb{R}^K)$ such that the following diagram is commutative:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}^k \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "\rho'"] & N \arrow[r] \arrow[d, "\rho"] & 1 \\ 1 \arrow[r] & \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k) \arrow[r] & \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k) \rtimes \mathrm{Aff}(\mathbb{R}^K) \arrow[r] & \mathrm{Aff}(\mathbb{R}^K) \arrow[r] & 1 \end{tikzcd}$$

If yes, clearly ρ' will be again canonical.

Crucial information about this iteration problem is contained in the connecting homomorphism δ of the long exact cohomology sequence

$$\to H^1(N, \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)/\mathbb{Z}^k) \xrightarrow{\delta} H^2(N, \mathbb{Z}^k) \to H^2(N, \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)) \to \tag{3}$$

according to the exact sequence of N-modules

$$0 \to \mathbb{Z}^k \to \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k) \to \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)/\mathbb{Z}^k \to 0 .$$

If E can be represented by a 2-cocycle f, with $\langle f \rangle \in H^2(N, \mathbb{Z}^k)$ lying in the image of δ, then the existence of an extended ρ' is true.

In [7] Nisse announced a very general proposition stating that δ is surjective, even in the case of noncentral extensions and more generally for polycyclic groups N. However, as shown in [5], a (solvable, not nilpotent) counter-example in the case of noncentral extensions casts doubt on this formulation. In the next section (see Example 4.11) we will show that Nisse's formulation is incorrect, even in the case of central extensions and for nilpotent groups N. Although the cases for which the iteration problem—for central extensions and nilpotent groups—seems to fail might be 'rare', their presence is complicating very strongly a universal treatment of the iteration problem.

As our interest at first was computational, we concentrated on constructing the representations rather than on proving their existence. Let us point out however, that the uniqueness of these representations (related to the eventual injectiveness of δ), has already been studied in [3].

Assume we are given a cohomology class $\langle f \rangle \in H^2(N, \mathbb{Z}^k)$, representing an extension $E = \mathbb{Z}^k \times N$ where the multiplication in E is given by

$$\forall z, z_1 \in \mathbb{Z}^k,\ \forall n, n_1 \in N: \quad (z, n)(z_1, n_1) = (z + z_1 + f(n, n_1), nn_1) .$$

If we can compute explicitly a 1-cochain, say $\gamma : N \to \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)$ killing the class $\langle f \rangle$ in $H^2(N, \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k))$, then $\rho'(z, n)$ is given by $\rho'(z, n) = (z + \gamma(n), \rho(n))$. As (2) shows, this extended representation will again be in canonical form.

This problem, in principle, now becomes a computational one. Indeed, we should find $\gamma : N \to \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)$, $x \mapsto \gamma(x) = (D(x), d(x))$, such that $\delta\gamma(x, y) = f(x, y)$ $(\forall x, y \in N)$. More explicitly, this means finding a matrix part $D(x)$ and a translational part $d(x)$, satisfying

$$^{x}(D(y), d(y)) - (D(xy), d(xy)) + (D(x), d(x)) = (0, f(x, y))$$

or equivalently

$$D(y)A(x^{-1}) - D(xy) + D(x) = 0\,, \tag{4}$$

$$D(y)(a(x^{-1})) + d(y) - d(xy) + d(x) = f(x, y)\,. \tag{5}$$

Since $\mathbb{Z}^k$ is a trivial N-module, this problem can be treated componentwise.

Now what looks like a 2-condition problem, surprisingly is a 1-condition problem, as we point out in the following proposition:

Proposition 3.1. *Assume $\rho : N \to \mathrm{Aff}(\mathbb{R}^K)$ is a representation in canonical form, and $1 \to \mathbb{Z} \to E \to N \to 1$ is a central extension, determined by $\langle f \rangle \in H^2(N, \mathbb{Z})$. Then, $\langle f \rangle$ lies in the image of $\delta : H^1(N, \mathrm{Aff}(\mathbb{R}^K, \mathbb{R})/\mathbb{Z}) \to H^2(N, \mathbb{Z})$ iff one can find $(D, d) : N \to \mathrm{Aff}(\mathbb{R}^K, \mathbb{R})$ satisfying condition* (5).

Proof. We will show that condition (4) is automatically satisfied, once condition (5) is fulfilled. So, assume (5) is satisfied. Since ρ is in canonical form, we know that the translational parts $a(x)$ (for $x \in N$) are spanning the whole vector space $\mathbb{R}^K$. Thus, it will be enough to show that

$$(D(y)A(x^{-1}) - D(xy) + D(x))a(z) = 0\,, \quad \forall z \in N\,. \tag{6}$$

Now, from $\rho(x^{-1}z) = \rho(x^{-1})\rho(z)$ it follows at once that, $a(x^{-1}z) = a(x^{-1}) + A(x^{-1}).a(z)$. This and the assumed relation (5) allows us to write (6) as

$$\begin{aligned} &D(y)(a(x^{-1}z) - a(x^{-1})) - D(xy)a(z) + D(x)a(z) \\ &\quad = f(z^{-1}x, y) - f(x, y) - f(z^{-1}, xy) + f(z^{-1}, x) \\ &\quad = -\delta f(z^{-1}, x, y) = 0\,. \end{aligned}$$

Note that we used both the fact that f is a 2-cocycle and that $\mathbb{Z}^k$ is considered as a trivial N-module. □

Remark 3.2. Assume, as above, $\langle f \rangle \in H^2(N, \mathbb{Z}^k)$. For a fixed embedding $\varepsilon : \mathbb{Z}^k \hookrightarrow \mathbb{R}^k$, $\langle f \rangle$ can also be identified with $\langle \varepsilon \circ f \rangle$ in $H^2(N, \mathbb{R}^k)$. Assume, $f = \delta(D, d)$ for $\delta : H^1(N, \mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)/\mathbb{Z}^k) \to H^2(N, \mathbb{Z}^k)$. Since $(D, d) = (D, 0) + (0, d)$ (where $(0, d)$ can be considered as a 1-cochain $N \to \mathbb{R}^k$) and $\delta(D, d) =$

$\delta(D,0)+\delta(0,d)$, it follows at once that, on the $\mathbb{R}^k$-level, $(\varepsilon \circ f) \simeq f'$ such that $f'=\delta(D,0)$.

4. Canonical form representations and matrices over polynomial rings

The examples we study later on, inspired us to detect an interesting property for canonical form affine representations. Basically, what we saw in all examples, were upper triangular matrices with polynomial entries and degrees going up towards the right upper corner of the matrix. We now prove this is what should happen.

For any commutative ring R with identity, we write $\mathbf{UT}_K(R)$ for the (multiplicative) group of upper-triangular $(K \times K)$-matrices with entries in R and 1's on the diagonal. A matrix A in $\mathbf{UT}_K(R)$ is called blocked upper triangular of type $(k_1, \ldots, k_c)$ (with $\sum_{i=1}^{c} k_i = K$) if and only if A has identity matrix blocks of size $k_1, \ldots, k_c$ on its diagonal. The subgroup of matrices in $\mathbf{UT}_K(R)$ which are of this type is denoted $\mathbf{BUT}_{\sum k_i}(R)$. From now on, we will speak of unitriangular and blocked unitriangular matrices.

The only possible nonzero entries (resp. blocks) in a matrix A of $\mathbf{UT}_K(R)$ (resp. $\mathbf{BUT}_{\sum k_i}(R)$) are the entries (resp. blocks) $a_{i,j}$ (resp. $A_{i,j} = (k_i \times k_j)$-block) with $j \geq i$. For these entries $a_{i,j}$ (resp. $A_{i,j}$) we call $(j-i)$ their distance from the diagonal. From now on we take R to be the ring $F[X_1, \ldots, X_m]$ of polynomials in m variables over a field F. We use this in the following:

Definition 4.1. A matrix A in $\mathbf{UT}_K(R)$ (resp. $\mathbf{BUT}_{\sum k_i}(R)$) is said to have the *Diagonal Distance Degree property* (*DDD-property*) if and only if each $a_{i,j}$ $(j>i)$ (resp. each entry in $A_{i,j}$ = the $(k_i \times k_j)$-block in A) is a polynomial of total degree $\leq (j-i)$. Such a matrix will be called a *DDD-matrix* (resp. a *blocked DDD-matrix of type* $\sum k_i$).

Lemma 4.2. *The set of all (blocked) DDD-matrices in* $\mathbf{UT}_K(R)$ *(resp.* $\mathbf{BUT}_{\sum k_i}(R)$*) forms a subgroup of* $\mathbf{UT}_K(R)$ *(resp.* $\mathbf{BUT}_{\sum k_i}(R)$*).* □

The proof is elementary and can be left to the reader.

Lemma 4.3. *Assume A is a fixed blocked unitriangular matrix of type $(k_1, \ldots, k_c)$ with entries in F. If $R = F[X]$, then there exists a DDD-matrix $B(X) \in \mathbf{BUT}_{\sum k_i}(R)$ such that $\forall l \in \mathbb{Z}$, $A^l = B(l)$.*

Proof. If $l \in \mathbb{Z}$, then clearly $A^l = ((A-I)+I)^l = \sum_{t=0}^{c} \binom{l}{t}(A-I)^t$. Evidently, A^l will be again unitriangular. If $j>i$, then

$$(A^l)_{i,j} = \left(\sum_{t=0}^{c} \binom{l}{t} (A-I)^t \right)_{i,j} = \sum_{t=0}^{j-i} \binom{l}{t} ((A-I)^t)_{i,j} ,$$

which is clearly seen to be of degree at most $(j-i)$ in l. So, it is sufficient to take $B(X) \in \mathbf{BUT}_{\Sigma k_i}(R)$ with

$$B(X)_{i,j} = \sum_{t=0}^{j-i} \binom{X}{t} ((A-I)^t)_{i,j} . \qquad \square$$

Remark 4.4. The upper-bound on the degree of the polynomials in $B(X)$ clearly depends also on the nilpotency degree of the matrix $(A-I)$, as is seen directly in the proof of the lemma. For example, if A is blocked unitriangular of type $\sum k_i$, and $(A-I)$ has m bottom rows of blocks which are zero, then $(A-I)^{c+1-m}=0$ and so the degree of the blocks $B(X)_{i,j}$ in $B(X)$ will be less than or equal to $\min\{j-i, c-m\}$.

Now, let us return to nilpotent groups. Assume a group N of rank $K = \sum_{i=1}^{c} k_i$. We describe a commutator presentation (of the type already given in (1)) in a more detailed manner. Let us label the generators with two indices to remember also which center in the upper central series of N they correspond with. That is, the generators $a_{1,1}, \ldots, a_{1,k_1}$ are the k_1 generators from the center $Z_1(N) = Z(N)$, $a_{2,1}, \ldots, a_{2,k_2}$ are the k_2 generators corresponding to $Z_2(N)/Z(N), \ldots$ and $a_{c,1}, \ldots, a_{c,k_c}$ are the k_c generators corresponding to $Z_c(N)/Z_{c-1}(N)$.

We consider the following way of ordering these labels: label (i, j) is said to be less than or equal to label (m, n) if and only if $m < i$ or $((m = i)$ and $(j \le n))$. Then the commutator presentation can be written as

$$\begin{aligned} &\langle a_{c,1}, a_{c,2}, \ldots, a_{c,k_c}, a_{c-1,1}, \ldots, a_{c-1,k_{c-1}}, \ldots, a_{1,1}, \ldots, a_{1,k_1} \mid \\ &[a_{i,t_j}, a_{m,t_n}] = \text{word in } a_{l,t_p}\text{'s}, (l, t_p) > (i, t_j) > (m, t_n) \rangle . \end{aligned} \tag{7}$$

A general element n of N is written in normal form as a word $n = a_{c,1}^{x_{c,1}} \ldots a_{1,k_1}^{x_{1,k_1}}$.

Regarding $\mathrm{Aff}(\mathbb{R}^K)$ as embedded in $\mathrm{Gl}(K+1, \mathbb{R})$ in the obvious way, we are ready for the following theorem:

Theorem 4.5. *Assume N a group of type $(k_1, \ldots, k_c)$ with a commutator presentation as in (7). Write $R = \mathbb{R}[x_{c,1}, \ldots, x_{1,k_1}]$. A representation $\rho : N \to \mathrm{Aff}(\mathbb{R}^K) \hookrightarrow \mathrm{Gl}(K+1, \mathbb{R})$ of N is in canonical form if and only if for $n = a_{c,1}^{x_{c,1}} \ldots a_{1,k_1}^{x_{1,k_1}} \in N$, $\rho(n)$ is a DDD-matrix in $\mathbf{BUT}_{(\Sigma k_i)+1}(R)$ combining the following properties:*

(1) *the total degree in the variables $(x_{i,1}, \ldots, x_{i,k_i})$ of the entries of $\rho(n)$ is less than or equal to i, more precisely: polynomial entries containing the variables $(x_{i,1}, \ldots, x_{i,k_i})$ occur*

- *in the linear part only in the blocks of the rth row, for $r \le i-1$, in terms of total degree at most $i-r$,*
- *or, in the translational part in the blocks of the rth row, for $r \le i$ in terms of total degree at most $i+1-r$; moreover, the ith block of the images of the generators $a_{i,t}$ $(1 \le t \le k_i)$ spans $\mathbb{R}^{k_i}$;*

(2) *if an entry* $a_{i,j}$ $(j>i)$ *in* $\rho(n)$ *is not zero, then it is a polynomial without constant term.*

Proof. The basic fact in the proof is given by the definition itself of a representation in canonical form. It follows immediately, that, for each i $(1\le i\le c)$ the generators $a_{i,j}$ $(1\le j\le k_i)$ are mapped by ρ to a matrix of the type

$$\begin{pmatrix} I_{k_1} & * & * & * & \dots & * & * \\ 0 & I_{k_2} & * & * & \dots & * & * \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & I_{k_i} & 0 & \dots & 0 & B_{i,j} \\ 0 & 0 & 0 & I_{k_{i+1}} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I_{k_c} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Here $B_{i,j}\in\mathbb{R}^{k_i}$ and $\mathbb{R}^{k_i}$ is spanned by $\{B_{i,1},B_{i,2},\dots,B_{i,k_i}\}$.

To finish the proof it is sufficient to realize that ρ is a homomorphism and to use the lemmas given above together with Remark 4.4. Then one proves successively that the following matrices satisfy the conditions (1) and (2) listed in the theorem:

(1) $\rho(a_{i,j}^{x_{i,j}})$,

(2) $\rho(a_{i,1}^{x_{i,1}}a_{i,2}^{x_{i,2}}\dots a_{i,k_i}^{x_{i,k_i}})$,

(3) $\rho(a_{c,1}^{x_{i,1}}\dots a_{m,j}^{x_{m,j}}\dots a_{1,k_1}^{x_{1,k_1}})$ (by induction on m).

The sufficiency of the conditions listed is also easily verified. □

Given N, there exists a unique simply connected, connected nilpotent Lie group G for which N is a uniform lattice. It is clear that the representation obtained in the theorem above, is also a Lie group representation for G. Indeed, for $\rho: N\to \mathrm{Aff}(\mathbb{R}^K)$ as before, we get a representation of G in $\mathrm{Aff}(\mathbb{R}^K)$ by allowing also reals to be substituted for the variables $x=(x_{1,1},\dots,x_{c,k_c})$.

Via exp and log this Lie group G is in one-to-one correspondence with its Lie algebra $\mathfrak{g}$. It becomes natural to ask for the meaning of canonical on the Lie algebra level. Therefore let us define the concept of a canonical representation of a nilpotent Lie algebra into $\mathfrak{aff}(\mathbb{R}^L)$, the semidirect product $\mathbb{R}^L\times\mathfrak{gl}(\mathbb{R}^L)$.

Assume $\mathfrak{g}$ is a nilpotent Lie algebra of dimension L and class c. Let $\mathfrak{g}_1=\mathfrak{g}$ and $\mathfrak{g}_{i+1}=\mathfrak{g}_i/Z(\mathfrak{g}_i)$, $i=1,\dots,c$. Let $\dim(Z(\mathfrak{g}_i))=l_i$ and write $L_i=\sum_{j\ge i} l_j$. $L=L_1=\dim(\mathfrak{g})$. So we have a series of epimorphisms:

$$\mathfrak{g}=\mathfrak{g}_1\xrightarrow{p_1}\mathfrak{g}_2\xrightarrow{p_2}\mathfrak{g}_3\twoheadrightarrow\cdots\twoheadrightarrow\mathfrak{g}_c\xrightarrow{p_c}\mathfrak{g}_{c+1}=0\,.$$

We write $Z_1(\mathfrak{g})=Z(\mathfrak{g})$ and $Z_i(\mathfrak{g})=(p_1^{-1}\circ\cdots\circ p_{i-1}^{-1})(Z(\mathfrak{g}_i))$ $(2\le i\le c)$.

Definition 4.6. A faithful (linear) representation $\rho:\mathfrak{g}\to\mathfrak{aff}(\mathbb{R}^K)$ will be called '*in canonical form*' (or a *canonical representation*) if and only if

(1) the matrix parts of ρ are blocked upper-triangular with the zero matrices of size $l_1, l_2, \ldots, l_c$ as diagonal entries,

(2) the ith block part of the subspace $Z_i(\mathfrak{g})$ consists of 'translations' ($\cong \mathbb{R}^{l_i}$) while higher-order blocks are all trivially zero.

Remark 4.7. We can choose a basis $A_{1,1}, \ldots, A_{1,l_1}, A_{2,1}, \ldots, A_{c,l_c}$ in such a way that $Z_i(\mathfrak{g})$ is the subalgebra of $\mathfrak{g}$ with basis $A_{1,1}, A_{1,2} \ldots A_{i,l_i}$. An embedding $\rho : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^L)$ is in canonical form, if and only if for an appropriate choice of such a basis

$$\rho(A_{i,j}) = \begin{pmatrix} O_{l_1} & * & * & * & \ldots & * & * \\ 0 & O_{l_2} & * & * & \ldots & * & * \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & O_{l_i} & 0 & \ldots & 0 & E_{i,j} \\ 0 & 0 & 0 & O_{l_{i+1}} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & O_{l_c} & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

where $E_{i,j} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{tr}}$ ($(l_i \times 1)$-matrix with 1 on the jth spot).

Use again $a_{1,1}, a_{1,2}, \ldots, a_{c,k_c}$ as the generators of N (7) and take $A_{i,j} = \log(a_{i,j})$. Now it is easily seen that $Z_i(\mathfrak{g})$ is an algebra spanned by $A_{1,1}, \ldots, A_{i,k_i}$. In particular, $\mathfrak{g}$ is spanned by all $A_{i,j}$'s. Assume $\rho : N \to \mathrm{Aff}(\mathbb{R}^K)$ canonical and write $A(x)$ for $\rho(n)$ where $n = a_{c,1}^{x_{c,1}} \ldots a_{1,k_1}^{x_{1,k_1}}$.

Theorem 4.8. *$\tilde{\rho} = \log \rho \exp$ is a linear representation of $\mathfrak{g}$ into $\mathfrak{aff}(\mathbb{R}^K)$, which is in canonical form and $\tilde{\rho}(x_{1,1}A_{1,1} + \cdots + x_{c,k_c}A_{c,k_c}) = B(x)$. Here, $B(x)$ is obtained from $A(x)$ by replacing all the diagonal identity-blocks by zero-blocks as well as by replacing all degree ≥ 2 parts of $A(x)$ by zero.*

Proof. $\tilde{\rho}$ ($= d\rho$, the differential of ρ) is a Lie algebra morphism, and so $\tilde{\rho}(x_{i,j}A_{i,j}) = x_{i,j}\tilde{\rho}(A_{i,j})$. This implies that the entries of this matrix, will be of degree 1 in the variable $x_{i,j}$.

On the other hand we see that for all $x_{i,j} \in \mathbb{Z}$, $\tilde{\rho}(x_{i,j}A_{i,j}) = \log(\rho(a_{i,j}^{x_{i,j}}))$. This means that $\log(\rho(a_{i,j}^{x_{i,j}}))$, as matrix, has degree-1 entries in the variable $x_{i,j}$.

We determine these entries by observing that

$$\log(\rho(a_{i,j}^{x_{i,j}})) = (\rho(a_{i,j}^{x_{i,j}}) - I) + \underbrace{\sum_{k=2}^{c} \frac{(-1)^{k+1}}{k} (\rho(a_{i,j}^{x_{i,j}}) - I)^k}_{\substack{\text{Containing only terms} \\ \text{of degree } \geq 2}} .$$

So the degree-1 terms come from $\rho(a_{i,j}^{x_{i,j}}) = A(x)|_{x=(0,\ldots,0,x_{i,j},0,\ldots,0)}$. This implies in turn that

$$\begin{aligned}\tilde{\rho}(x_{1,1}A_{1,1} + \cdots + x_{c,k_c}A_{c,k_c})\\ = x_{1,1}\log(\rho(a_{1,1})) + \cdots + x_{c,k_c}\log(\rho(a_{c,k_c})) = B(x)\,.\end{aligned}$$

Now, by looking at the matrices of $\tilde{\rho}(A_{i,j})$, one concludes that $\tilde{\rho}$ is in canonical form. □

Conversely, if one considers a canonical embedding $\tilde{\rho}$ of $\mathfrak{g}$ into $\mathfrak{aff}(\mathbb{R}^K)$ then one can easily see that $\rho = \exp\tilde{\rho}\log : G \to \mathrm{Aff}(\mathbb{R}^K)$ induces a representation of N, which is in canonical form.

Now that we have a better picture of canonical type representations, we want to come back to the iteration problem as stated in the previous section. Also we can point out here that there has already been some interest in the literature for similar-looking iterative work concerning complete normal Koszul–Vinberg (KV-) structures on nilpotent Lie algebras [1]. Note that although a canonical Lie algebra representation determines a complete KV-structure, this KV-structure will not necessarily be normal.

The following example, communicated to us on the Lie algebra level by Dan Segal and Fritz Grunewald (to whom we express our gratitude), shows however that a great amount of care will be necessary with respect to the 'universal' nature of both iterative approaches.

For clarity, we prefer to present the example twice: once on the Lie algebra level and once on the group level. As a consequence it will follow that

(1) The final theorem, called the 'Lifting theorem', in [1], is incorrect as stated there.

(2) The iteration problem as stated previously does generally not have a positive answer; a fortiori the announcement of a positive answer to a much more general version in [7] is incorrect.

(3) One should pay attention to have a good understanding of the 3-step nilpotent case in [4].

Example 4.9 (Lie algebra level). To permit the reader an easier comparison with the situation in [1], we use the notations and terminology adopted there. Consider the 4-dimensional 3-step nilpotent Lie algebra $\mathfrak{g} = \langle A_1, A_2, A_3, A_4\rangle$ where the brackets are defined by

$$[A_1, A_2] = A_3\,, \quad [A_1, A_3] = A_4\,, \quad [A_2, A_3] = 0 = [\mathfrak{g}, A_4]\,.$$

It is easily seen that $\bar{\mathfrak{g}} = \mathfrak{g}/\langle A_4\rangle = \langle \bar{A}_1, \bar{A}_2, \bar{A}_3\rangle$ is the Heisenberg algebra. In $\bar{\mathfrak{g}}$, let us consider the flag of ideals $F(\bar{\mathfrak{g}})$, given by

$$F(\bar{\mathfrak{g}}):\quad \bar{\mathfrak{g}} = \bar{\mathfrak{g}}_3 \supset \bar{\mathfrak{g}}_2 = \langle \bar{A}_1, \bar{A}_3\rangle \supset \bar{\mathfrak{g}}_1 = \langle \bar{A}_3\rangle \supset 0\,.$$

Remark that this flag is finer than the lower central series of $\bar{\mathfrak{g}}$. It is not hard to verify that the following linear representation $\tilde{\rho}$ of $\bar{\mathfrak{g}}$ is a complete, normal Koszul–Vinberg structure (KV-structure); $\tilde{\rho}$ is defined by

$$\tilde{\rho}(\bar{A}_2)\bar{A}_1 = \bar{A}_3\,, \quad \tilde{\rho}(\bar{A}_1)\bar{A}_2 = 2\bar{A}_3\,,$$
$$\tilde{\rho}(\bar{A}_i)\bar{A}_j = 0 \quad \text{in all other cases}\,.$$

We now proceed to show that this KV-structure does not lift to a normal KV-structure ρ on $\mathfrak{g}$. Taking into account the lower central series of $\mathfrak{g}$, one verifies that a normal KV-structure ρ on $\mathfrak{g}$ must satisfy

$$\rho(A_i)A_4 = 0\,, \quad \rho(A_3)A_3 = 0\,, \quad \rho(\mathfrak{g})\mathfrak{g} = \langle A_3, A_4\rangle\,.$$

Furthermore, a lifting of $\tilde{\rho}$ must have at least the following properties:

$$\rho(A_2)A_1 = A_3 + \alpha A_4\,, \qquad \rho(A_1)A_1 = \gamma A_4\,,$$
$$\rho(A_1)A_3 = \beta A_4\,, \qquad \rho(A_3)A_1 = (\beta - 1)A_4\,.$$

However, from the definition of KV-structure in [1] it follows that we should also have

$$\rho(A_2)\rho(A_1)A_1 - \rho(A_1)\rho(A_2)A_1 = \rho([A_2, A_1])A_1$$
$$\Rightarrow \quad -\beta A_4 = (1 - \beta)A_4$$

and this is clearly a contradiction.

Remark 4.10. This situation is 'exceptional', because every complete, normal KV-structure $(k \neq 0)$

$$\tilde{\rho}(\bar{A}_2)\bar{A}_1 = k\bar{A}_3\,, \quad \tilde{\rho}(\bar{A}_1)\bar{A}_2 = (k+1)\bar{A}_3\,,$$
$$\tilde{\rho}(\bar{A}_i)\bar{A}_j = 0 \quad \text{in all other cases}$$

on $\bar{\mathfrak{g}}$, with $k \neq 1$, extends to a normal KV-structure on $\mathfrak{g}$.

Example 4.11 (group level). We now reconsider this example on the group level. Take the 3-step nilpotent group:

$$N_3\colon \quad \langle a_1, a_2, a_3, a_4 \mid [a_2, a_1] = a_3^{-1}, [a_3, a_1] = a_4\rangle\,.$$

Although this group can be given a canonical representation as will be described in Section 6, we show that it can serve as a critical example with respect to the iteration problem.

N_3 can be seen as a central extension of

$$N_2\colon \langle a_1, a_2, a_3 \mid [a_2, a_1] = a_3^{-1} \rangle .$$

Consider the following canonical representation ($\tilde{\rho}$) of N_2:

$$\tilde{\rho}(a_1^{x_1} a_2^{x_2} a_3^{x_3}) = \begin{pmatrix} A(x) & a(x) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_2 & 2x_1 & 2x_1x_2 + x_3 \\ 0 & 1 & 0 & x_1 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

We show that $\tilde{\rho}$ cannot be lifted to N_3. Lemma 6.8 will show that N_3 is determined by a cocycle $f(x, y) = x_3 y_1 - x_2\binom{y_1}{2}$. Suppose $\gamma = (D, d) : N_2 \to \mathrm{Aff}(\mathbb{R}^3, \mathbb{R}^1)$ satisfies equations (4) and (5). Take

$$\gamma(a_1) = ((\alpha_1, \alpha_2, \alpha_3), \alpha_4) , \quad \gamma(a_2) = ((\beta_1, \beta_2, \beta_3), \beta_4) ,$$
$$\gamma(a_3) = ((\varepsilon_1, \varepsilon_2, \varepsilon_3,), \varepsilon_4) ,$$

then

$$0 = D(a_3)A(a_3) - D(a_3^{-1}a_3) + D(a_3^{-1}) \quad \Rightarrow \quad D(a_3^{-1}) = -D(a_3)$$

and

$$0 = D(a_2)A(a_1^{-1}) - D(a_1a_2) + D(a_1)$$
$$\Rightarrow \quad D(a_1a_2) = (\beta_1 + \alpha_1, \beta_2 + \alpha_2, -2\beta_1 + \beta_3 + \alpha_3) .$$

Since $a_2a_1 = a_1a_2a_3^{-1}$ (in N_2)

$$D(a_2a_1) = D(a_1a_2a_3^{-1}) = D(a_3^{-1})A(a_2^{-1}a_1^{-1}) + D(a_1a_2) .$$

Use this in

$$0 = D(a_1)A(a_2^{-1}) - D(a_2a_1) + D(a_2)$$

to find that $\varepsilon_2 = \alpha_1$.

On the other hand, since

$$f(a_1, a_3) = 0 = D(a_3)a(a_1^{-1}) + d(a_3) - d(a_1a_3) + d(a_1)$$

it follows that

$$d(a_1a_3) = d(a_3a_1) = -\varepsilon_2 + \varepsilon_4 + \alpha_4 .$$

However, from

$$f(a_3, a_1) = 1 = D(a_1)a(a_3^{-1}) + d(a_1) - d(a_3a_1) + d(a_3)$$

it follows that $\varepsilon_2 = \alpha_1 + 1$ and we have a contradiction. Therefore, the morphism

$$j : H^2(N_2, \mathbb{Z}) \rightarrow H^2(N_2, \mathrm{Aff}(\mathbb{R}^3, \mathbb{R}^1))$$

is nonzero for this particular choice of $\tilde{\rho}$. □

The careful reader should note that this does not contradict Theorem 2.5 in [4]. Indeed, N_3 does have a canonical representation; however, this representation has to take off at the 2-step level with a canonical representation which is different from $\tilde{\rho}$ above. In other words, Theorem 2.5 in [4] should not be understood as the vanishing of the morphism

$$j : H^2(N_2, \mathbb{Z}^k) \rightarrow H^2(N_2, \mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^k))$$

for whatever canonical type representation of N_2.

Rather than stating evidence for the truth of the iteration problem in its universal setting as above, our computational treatment of the examples in the following sections gives evidence for the existence of canonical type representations for torsion free, finitely generated nilpotent groups. Therefore, we conclude this section with the following:

The iteration problem (restricted setting). 'Every torsion free, finitely generated nilpotent group has a canonical type affine representation.'

5. 2-step nilpotent groups

Every torsion free, finitely generated 2-step nilpotent group N can be given a commutator presentation as follows

$$\begin{aligned} N: \quad & \langle a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+k} \mid \\ & [a_i, a_j] = a_{n+1}^{l_{1,i,j}} a_{n+2}^{l_{2,i,j}} \ldots a_{n+k}^{l_{k,ij}} \; (1 \le j < i \le n) \rangle \,. \end{aligned} \tag{8}$$

Identify an element $n \in N$ is with its coordinate vector $x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})$.

As has been indicated in [2], N can be represented faithfully and in canonical form, as a group of affine transformations of $\mathbb{R}^{n+k}$. As an example and in view of further use, we give the explicit representation obtained for N as above. $\rho : N \to \mathrm{Aff}(\mathbb{R}^{n+k})$ is defined as follows: let $n \in N$, with coordinates $(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k})$; then

$\rho(n) = (\mathrm{Lin}(n), \mathrm{Tr}(n)) =$

$$\left(\begin{array}{cccccccccc}
1 & 0 & \dots & 0 & 0 & -x_1 l_{1,2,1} & -\sum_{t=1}^{2} x_t l_{1,3,t} & \dots & -\sum_{t=1}^{n-1} x_t l_{1,n,t} \\
0 & 1 & \dots & 0 & 0 & -x_1 l_{2,2,1} & -\sum_{t=1}^{2} x_t l_{2,3,t} & \dots & -\sum_{t=1}^{n-1} x_t l_{2,n,t} \\
\vdots & & & & \vdots & & & & \\
0 & 0 & \dots & 1 & 0 & -x_1 l_{k,2,1} & -\sum_{t=1}^{2} x_t l_{k,3,t} & \dots & -\sum_{t=1}^{n-1} x_t l_{k,n,t} \\
0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\
\vdots & & & & \vdots & & & & \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1
\end{array}\right)
\left(\begin{array}{c}
x_{n+1} - \sum_{1 \le t < s \le n} x_t x_s l_{1,s,t} \\
x_{n+2} - \sum_{1 \le t < s \le n} x_t x_s l_{2,s,t} \\
\vdots \\
x_{n+k} - \sum_{1 \le t < s \le n} x_t x_s l_{k,s,t} \\
x_1 \\
x_2 \\
\vdots \\
x_n
\end{array}\right) \tag{9}$$

6. 3-step nilpotent groups of type $(k_1, k_2, 2)$

In this section we describe canonical form representations for all 3-step nilpotent groups N_3 of type $(k_1, k_2, 2)$. As indicated earlier, it will be enough to construct these representations for groups of type $(1, k_2, 2)$.

Our approach towards these examples is based on writing down general commutator presentations and treating these in the iterative way previously described.

A 2-step nilpotent group, N_2, of rank n and having a center of rank $n-2$ (i.e. of type $(k_2 = n-2, 2)$) can be given a commutator presentation

$$N_2\colon \quad \langle a_1, a_2, \dots, a_n \mid [a_2, a_1] = a_3^{\alpha_3} a_3^{\alpha_4} \dots a_n^{\alpha_n} \rangle .$$

It is always possible to rearrange the generators of N_2 so that $\alpha_3 \neq 0$ (notice, that if all α's are zero, the group becomes $\mathbb{Z}^n$).

Now, consider a central extension

$$0 \to \mathbb{Z} \to N_3 \to N_2 \to 1 \tag{10}$$

in order to obtain a 3-step nilpotent group N_3 with a presentation

$$\begin{aligned} N_3\colon \quad \langle a_1, a_2, \dots, a_{n+1} \mid [a_2, a_1] &= a_3^{\alpha_3} a_4^{\alpha_4} \dots a_n^{\alpha_n} a_{n+1}^{k_{2,1}} , \\ [a_i, a_j] &= a_{n+1}^{k_{i,j}} \ (\text{for } n \ge i > j \ge 1, (i, j) \neq (2, 1)) \rangle . \end{aligned} \tag{11}$$

It is important to note, as the following example demonstrates, that one cannot hope to choose the $k_{i,j}$'s independently from each other.

Example 6.1. Take the following presentation which is clearly of type (11):

$$\langle a_1, a_2, a_3, a_4, a_5 \mid [a_2, a_1] = a_3 a_4, [a_4, a_3] = a_5 \rangle .$$

Since all commutators which are not explicitly specified, are supposed to be trivial we can easily verify that

$$a_2 a_1 = a_3^{-1}(a_2 a_1) a_3 = a_3^{-1}(a_1 a_2 a_3 a_4) a_3 = a_1 a_2 a_3 a_4 a_5 = a_2 a_1 a_5$$

from which it follows that $a_5 = 1$. We really wanted the above group to be torsion free finitely generated 3-step nilpotent of rank 5. This 'incompatibility' shows this is not the case.

The example shows that in a general presentation of type (11), we should not accept all possible values for the $k_{i,j}$'s. Since we want our presentation to be '*computational consistent*' we have to take into account consistency conditions.

Indeed, it follows at once from (11) that $a_2 a_1 = a_1 a_2 a_3^{\alpha_3} a_4^{\alpha_4} \ldots a_n^{\alpha_n} a_{n+1}^{k_{2,1}}$. Now, take $r \geq 3$. Consequently we should have

$$\begin{aligned}
& a_r a_2 a_1 \\
&= a_2 a_1 a_r a_{n+1}^{k_{r,1} + k_{r,2}} \\
&= a_1 a_2 a_3^{\alpha_3} a_4^{\alpha_4} \ldots a_n^{\alpha_n} a_r a_{n+1}^{k_{2,1} + k_{r,1} + k_{r,2}} \\
&= a_1 a_2 a_r a_3^{\alpha_3} \ldots a_n^{\alpha_n} a_{n+1}^{k_{2,1} + k_{r,1} + k_{r,2} - \alpha_3 k_{r,3} - \cdots - \alpha_{r-1} k_{r,r-1} + \alpha_{r+1} k_{r+1,r} + \cdots + \alpha_n k_{n,r}} \\
&= a_r a_1 a_2 a_3^{\alpha_3} \ldots a_n^{\alpha_n} \alpha_{n+1}^{k_{2,1} - \alpha_3 k_{r,3} - \cdots - \alpha_{r-1} k_{r,r-1} + \alpha_{r+1} k_{r+1,r} + \cdots + \alpha_n k_{n,r}} .
\end{aligned}$$

Since this happens, for each $r \geq 3$, we get a system of $(n-2)$ equations linking the constants $k_{i,j}$ to each other. So, we have shown the following lemma:

Lemma 6.2. *In order to let the presentation* (11) *define a torsion free, nilpotent group of rank* $n+1$ *the following consistency conditions should be fulfilled for the constants* $k_{i,j}$*'s*:

$$\begin{cases}
\alpha_4 k_{4,3} + \alpha_5 k_{5,3} + \alpha_6 k_{6,3} + \cdots + \alpha_n k_{n,3} = 0 , \\
-\alpha_3 k_{4,3} + \alpha_5 k_{5,4} + \alpha_6 k_{6,4} + \cdots + \alpha_n k_{n,4} = 0 , \\
\quad \vdots \\
-\alpha_3 k_{r,3} - \cdots - \alpha_{r-1} k_{r,r-1} + \alpha_{r+1} k_{r+1,r} + \cdots + \alpha_n k_{n,r} = 0 , \\
\quad \vdots \\
-\alpha_3 k_{n,3} - \alpha_4 k_{n,4} - \alpha_5 k_{n,5} - \cdots - \alpha_{n-1} k_{n,n-1} = 0 . \qquad \square
\end{cases}$$

Remark 6.3. As will turn out these necessary conditions will be also sufficient. This follows from the fact that, under the conditions of the lemma, the groups N_3 can be embedded in canonical form, which conversely, guarantees their desired properties.

Now in order to build an affine representation in canonical form for the groups N_3 given by a presentation (11), we recall the basic steps:

Via the embedding (9) N_2 has a representation $\rho : N_2 \to \mathrm{Aff}(\mathbb{R}^n)$ which is in canonical form. If we write x for elements of N_2, meaning $a_1^{x_1} \dots a_n^{x_n}$, we have

$$\rho(x) = \left[\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & -\alpha_3 x_1 \\ 0 & 1 & \dots & 0 & 0 & -\alpha_4 x_1 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 & -\alpha_n x_1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} x_3 - \alpha_3 x_1 x_2 \\ x_4 - \alpha_4 x_1 x_2 \\ \vdots \\ x_n - \alpha_n x_1 x_2 \\ x_1 \\ x_2 \end{pmatrix} \right] . \tag{12}$$

What we need to do (as described in Section 3) is to determine a cocycle describing the extension (10) and to solve a system of equations as given in (4) and (5). Now, in order to do this in complete generality, it turned out to be adequate to divide the work in two cases which will be combined later.

Case 1: In this case we consider groups N_3 with a presentation

$$\begin{aligned} N_3\colon\quad \langle a_1, a_2, \dots, a_{n+1} \mid [a_2, a_1] &= a_3^{\alpha_3} a_4^{\alpha_4} \dots a_n^{\alpha_n} , \\ [a_i, a_j] &= a_{n+1}^{k_{i,j}} \ (\text{for } i > j \geq 3) \rangle . \end{aligned} \tag{13}$$

Case 2: Here we mean groups N_3 with a presentation

$$\begin{aligned} N_3\colon\quad \langle a_1, a_2, \dots, a_{n+1} \mid [a_2, a_1] &= a_3^{\alpha_3} a_4^{\alpha_4} \dots a_n^{\alpha_n} a_{n+1}^{k_{2,1}} , \\ [a_i, a_1] &= a_{n+1}^{k_{i,1}} \ (\text{for } i \geq 3) , \\ [a_i, a_2] &= a_{n+1}^{k_{i,2}} \ (\text{for } i \geq 3) \rangle . \end{aligned} \tag{14}$$

6.1. The representation for Case 1

Fix a presentation of type (13). Taking a section $s : N_2 \to N_3$, $x \mapsto a_1^{x_1} \dots a_n^{x_n}$ in (10), we compare $s(x).s(y)$ with $s(x \cdot y)$ to obtain the corresponding 2-cocycle $f : N_2 \times N_2 \to \mathbb{Z}$. We do this for positive values of the x_i and y_j; this is sufficient since we know that this cocycle is a polynomial which will be determined completely by knowing its values over the positive integers.

Let us note first of all that (in N_3), we find

$$\begin{aligned} a_2^{x_2} a_1^{y_1} &= a_1^{y_1} a_2^{x_2} (a_3^{\alpha_3} \dots a_n^{\alpha_n}) \dots (a_3^{\alpha_3} \dots a_n^{\alpha_n}) \quad (x_2 y_1 \text{ times } (a_3^{\alpha_3} \dots a_n^{\alpha_n})) \\ &= a_1^{y_1} a_2^{x_2} a_3^{\alpha_3 x_2 y_1} \dots a_n^{\alpha_n x_2 y_1} a_{n+1}^{\binom{x_2 y_1}{2} \sum_{i > j \geq 3} \alpha_i \alpha_j k_{i,j}} . \end{aligned} \tag{15}$$

This is used in

$$
\begin{aligned}
s(x).s(y) &= a_1^{x_1} a_2^{x_2} a_1^{y_1} a_2^{y_2} a_3^{x_3+y_3} \dots a_n^{x_n+y_n} a_{n+1}^{\Sigma_{i>j\geq 3} k_{i,j} x_i y_j} \\
&\overset{(15)}{=} a_1^{x_1+y_1} a_2^{x_2+y_2} a_3^{\alpha_3 x_2 y_1} \dots a_n^{\alpha_n x_2 y_1} a_3^{x_3+y_3} \dots \\
&\quad a_n^{x_n+y_n} a_{n+1}^{\Sigma_{i>j\geq 3}(k_{i,j} x_i y_j + \binom{x_2 y_1}{2} \alpha_i \alpha_j k_{i,j})}
\end{aligned}
$$

from which a cocycle expression for (10) now easily follows. We given this in the following lemma:

Lemma 6.4. *An extension $0 \to \mathbb{Z} \to N_3 \to N_2 \to 1$ determining a group N_3 with a presentation of type* (13) *can be described via a (polynomial) cocycle $f : N_2 \times N_2 \to \mathbb{Z}$ given by*

$$
\begin{aligned}
f(x, y) = {} & \sum_{i>j\geq 3} k_{i,j} x_i y_j + \binom{x_2 y_1}{2} \sum_{i>j\geq 3} \alpha_i \alpha_j k_{i,j} \\
& + x_2 y_1 \sum_{j>i\geq 3} (x_i + y_i) \alpha_j k_{k,i} \,.
\end{aligned}
\tag{16}
$$

Now, constructing an affine representation of N_3 into $\mathrm{Aff}(\mathbb{R}^{n+1})$, reduces to constructing a 1-cochain $\gamma : N_2 \to \mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^1)$ so that our cocycle f equals the coborder $\partial\gamma$. Define

$$
\gamma : N_2 \to \mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^1)\,, \qquad x \mapsto \gamma(x) = (\mathrm{Lin}(\gamma(x)), \mathrm{Tr}(\gamma(x)))\,, \tag{17}
$$

where the linear part is taken to be

$$
\mathrm{Lin}(\gamma(x)) = \left(\sum_{i=3}^{n} A_{i,3} x_i, \sum_{i=3}^{n} A_{i,4} x_i, \dots, \sum_{i=3}^{n} A_{i,n} x_i, 0, 0 \right) \tag{18}
$$

and the translational part is given by

$$
\mathrm{Tr}(\gamma(x)) = T x_3 - \sum_{i=3}^{n} A_{i,i} \binom{x_i}{2} - \sum_{\substack{j=3 \\ i>j}}^{n} A_{i,j} x_i x_j \,. \tag{19}
$$

We now describe a way, quite easy to carry out, to determine the $A_{i,j}$'s and T such that $\partial\gamma = f$. We proceed as follows:

Step 1: Choose $A_{k,l}$ $(4 \leq k,l \leq n)$ in such a way that

$$
A_{i,j} - A_{j,i} = k_{i,j} \quad \text{for } 4 \leq j < i \leq n \,.
$$

For example we could make the following choice:

Step 1.a: Take $A_{i,j}=0$, for $4\le i\le j\le n$.

Step 1.b: Take $A_{i,j}=k_{i,j}$ for $4\le j<i\le n$.

To get a good picture of what is going on we can organize the $A_{i,j}$'s in an $(n-2)\times(n-2)$-matrix where $A_{3,3}$ takes the upper left corner position. In the example suggested the matrix $(A_{i,j})$ can be pictured as follows

$$(A_{i,j})_{3\le i\le n, 3\le j\le n}=\begin{pmatrix} * & * & * & \dots & * & * \\ * & 0 & 0 & \dots & 0 & 0 \\ * & k_{5,4} & 0 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ * & k_{n,4} & k_{n,5} & \dots & k_{n,n-1} & 0 \end{pmatrix}. \tag{20}$$

Step 2: In the above matrix, we now determine the terms on the first row and the first column such that the following conditions are satisfied:

$$\sum_{i=3}^{n}\alpha_i A_{i,j}=0 \quad \text{for } 3\le j\le n \ \ (\text{this determines } A_{3,j}\ (j>3))\,, \tag{21}$$

$$\sum_{j=3}^{n}\alpha_j A_{i,j}=0 \quad \text{for } 3\le i\le n \ \ (\text{this determines } A_{i,3}\ (i>3))\,. \tag{22}$$

At this point, the careful reader will notice that the upper left corner of the matrix $A_{i,j}$ can be chosen to fulfil two conditions: on the one hand, following (22) we have

$$A_{3,3}=-\frac{1}{\alpha_3}(\alpha_4A_{3,4}+\cdots+\alpha_nA_{3,n})$$

and on the other hand, following (21) we find

$$A'_{3,3}=-\frac{1}{\alpha_3}(\alpha_4A_{4,3}+\cdots+\alpha_nA_{n,3})\,.$$

The following lemma shows that this double interpretation causes no problems.

Lemma 6.5. *For all* $r>3$ *we have*

$$A_{r,3}-A_{3,r}=k_{r,3}\,.$$

In particular, it follows that $A'_{3,3}=A_{3,3}$.

Proof. For $r>3$ we have

$$A_{r,3}=-\frac{1}{\alpha_3}(\alpha_4A_{r,4}+\cdots+\alpha_nA_{r,n}) \quad (\text{from (22)})$$

and

$$A_{3,r} = -\frac{1}{\alpha_3}(\alpha_4 A_{4,r} + \cdots + \alpha_n A_{n,r}) \quad \text{(from (21))}.$$

So, we get

$$\begin{aligned} A_{r,3} - A_{3,r} &= \frac{1}{\alpha_3}(\alpha_4(A_{4,r} - A_{r,4}) + \cdots + \alpha_n(A_{n,r} - A_{r,n})) \\ &= \frac{1}{\alpha_3}(-\alpha_4(A_{r,4} - A_{4,r}) - \cdots - \alpha_{r-1}(A_{r,r-1} - A_{r-1,r}) \\ &\qquad + \alpha_{r+1}(A_{r+1,r} - A_{r,r+1}) + \cdots + \alpha_n(A_{n,r} - A_{r,n})) \\ &= k_{r,3} \quad \text{(by Lemma 6.2)}. \end{aligned}$$

This now implies

$$\begin{aligned} \alpha_3(A_{3,3} - A'_{3,3}) &= \alpha_4(A_{4,3} - A_{3,4}) + \cdots + \alpha_n(A_{n,3} - A_{3,n}) \\ &= \alpha_4 k_{4,3} + \cdots + \alpha_n k_{n,3} = 0. \qquad \square \end{aligned}$$

Remark 6.6. It is quite natural to consider $k_{i,i} = 0$ $(i \geq 3)$ and to put $k_{i,j} = -k_{j,i}$ (for $i < j$). It then follows easily that we always have $A_{i,j} - A_{j,i} = k_{i,j}$.

We are now ready to state the following proposition:

Proposition 6.7. *For an extension* (10) *determining a group* N_3 *with a presentation* (13), *and described by a* 2-*cocycle* (16) f, *and for the* 1-*cochain* $\gamma : N_2 \to \mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^1)$ *defined as in* (17) *it suffices to use the* $A_{i,j}$*'s as constructed above and to take*

$$T = \frac{1}{\alpha_3}\left(\sum_{\substack{j>i \\ i=3}}^{n} A_{j,i}\alpha_i\alpha_j + \frac{1}{2}\sum_{i=3}^{n} A_{i,i}\alpha_i(\alpha_i - 1)\right) \tag{23}$$

to obtain $\partial\gamma = f$.

Proof. We recall that $\partial\gamma(x, y) = {}^x\gamma(y) - \gamma(x * y) + \gamma(x)$. Let us first of all verify that

$$\sigma(x^{-1}) = \left[\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \alpha_3 x_1 \\ 0 & 1 & \dots & 0 & 0 & \alpha_4 x_1 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 & \alpha_n x_1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -x_3 \\ -x_4 \\ \vdots \\ -x_n \\ -x_1 \\ -x_2 \end{pmatrix}\right].$$

Now compute

$$\begin{aligned}{}^{x}\gamma(y) &= \gamma(y)\circ\sigma(x^{-1})\\ &= (\mathrm{Lin}(\gamma(y))\cdot\mathrm{Lin}(\sigma(x^{-1})), \mathrm{Lin}(\gamma(y)).\mathrm{Tr}(\sigma(x^{-1})) + \mathrm{Tr}(\gamma(y)))\,.\end{aligned}$$

Referring to (4) and (5) the proof finishes in principle by verifying that $\mathrm{Lin}(\partial\gamma(x, y)) = (0, 0, \ldots, 0)$ (which is left to the reader) and that, for an appropriate choice of T in (19) $\mathrm{Tr}(\partial\gamma(x, y)) = f(x, y)$ (16). The verification of this translational part requires careful work. We reduced it to the following important intermediate steps:

One begins with verifying that

$$\begin{aligned}\mathrm{Tr}(\partial\gamma(x, y)) = {} & \sum_{\substack{i,j=3\\ j<i}}^{n} (A_{i,j} - A_{j,i})x_i y_j - T\alpha_3 x_2 y_1 - \sum_{i=3}^{n} \frac{A_{i,i}\alpha_i}{2}\, x_2 y_1\\ & + \sum_{i=3}^{n} \left(\frac{A_{i,i}\alpha_i^2}{2}\,(x_2y_1)^2 + A_{i,i}\alpha_i x_2 y_1(x_i + y_i)\right)\\ & + \sum_{\substack{i=3\\ j>i}}^{n} A_{j,i}\alpha_i\alpha_j(x_2y_1)^2\\ & + \sum_{\substack{i=3\\ j>i}}^{n} (A_{j,i}\alpha_i x_2 y_1(x_j + y_j) + A_{j,i}\alpha_j x_2 y_1(x_i + y_i))\,. \qquad (24)\end{aligned}$$

In this expansion we now put

$$T = \frac{1}{\alpha_3}\left(\sum_{\substack{i=3\\ j>i}}^{n} A_{j,i}\alpha_i\alpha_j + \sum_{i=3}^{n} \frac{A_{i,i}\alpha_i(\alpha_i - 1)}{2}\right).$$

Taking into account Remark 6.6, we can now transform (24) to obtain

$$\begin{aligned}\mathrm{Tr}(\partial\gamma(x, y)) = {} & \sum_{\substack{i=3\\ i>j}}^{n} k_{i,j}x_i y_j\\ & + \left(\frac{x_2y_1}{2}\right)\left(\sum_{i=3}^{n} \alpha_i\left(\sum_{j>i} A_{j,i}\alpha_j + \sum_{j<i} A_{i,j}\alpha_j + A_{i,i}\alpha_i\right)\right)\\ & + \sum_{i=3}^{n} x_2 y_1(x_i + y_i)\left(\sum_{j>i} A_{j,i}\alpha_j + \sum_{j<i} A_{i,j}\alpha_j + A_{i,i}\alpha_i\right)\end{aligned}$$

Here we find $\sum_{j>i} A_{j,i}\alpha_j + \sum_{j<i} A_{i,j}\alpha_j + A_{i,i}\alpha_i$ twice. One verifies that this equals $\sum_{j>i} k_{j,i}\alpha_j$.

Now, we conclude with

$$\operatorname{Tr}(\partial\gamma(x, y)) = \sum_{i>j\geq 3} k_{i,j}x_i y_j + \binom{x_2 y_1}{2} \sum_{i>j\geq 3} k_{i,j}\alpha_i\alpha_j + x_2 y_1 \Big(\sum_{j>i\geq 3} (x_i + y_i)\alpha_j k_{j,i} \Big) \tag{25}$$

which is seen to be the cocycle f. □

As an example later on will show, it is fairly easy to construct the affine presentation for extended group N_3, once an explicit formula for the cochain γ as above is known.

6.2. The representation for Case 2

Here we take a presentation of type (14) to start with. When seen as arising from an extension $1 \to \mathbb{Z} \to N_3 \to N_2 \to 1$ as before, once again we compute a cocycle for this extension.

We proceed as before and take a section $s: N_2 \to N_3$, $x \mapsto a_1^{x_1} \ldots a_n^{x_n}$. It is quite trivial that the cocycle f, defined from $s(x).s(y) = s(x.y).f(x, y)$, and which is a polynomial in the x_i and y_j, is linear in the $k_{i,j}$.

Let us denote with $f_{i,j}$ the cocycle corresponding to the extension above in the case where $k_{i,j} = 1$ and $k_{l,m} = 0$ (for $(l, m) \neq (i, j)$). It will then follow easily that

$$f(x, y) = \sum_{i>j\geq 1}^{n} k_{i,j} f_{i,j}(x, y) .$$

It is immediately verified that $f_{2,1}(x, y) = x_2 y_1$.

Now, we look at $f_{i,1}$ for $i > 2$: in

$$s(x).s(y) = a_1^{x_1} a_2^{x_2} a_1^{y_1} a_2^{y_2} a_3^{x_3+y_3} \ldots a_n^{x_n+y_n} a_{n+1}^{x_i y_1}$$

we have special attention for $a_2^{x_2} a_1^{y_1}$.

It is easily seen that

$$a_2^{x_2} a_1^{y_1} = a_1^{y_1} a_2^{x_2} a_3^{x_2 y_1 \alpha_3} \ldots a_n^{x_2 y_1 \alpha_n} a_{n+1}^{\binom{y_1}{2}\alpha_i x_2}$$

so that we can conclude with

$$f_{i,1}(x, y) = \alpha_i \binom{y_1}{2} x_2 + x_i y_1 \quad \text{for } i > 2 .$$

Analogously, we compute $f_{i,2}$ for $i > 2$ and here we find

$$f_{i,2}(x, y) = \alpha_i y_1 y_2 x_2 + \binom{x_2}{2} \alpha_i y_1 + x_i y_2 .$$

These computations result in the following lemma:

Lemma 6.8. *An extension* $0 \to \mathbb{Z} \to N_3 \to N_2 \to 1$ *determining a group* N_3 *with a presentation of type* (14) *can be described via a cocycle* $f : N_2 \times N_2 \to \mathbb{Z}$ *given by*

$$f(x, y) = k_{2,1} x_2 y_1 + \sum_{i=3}^{n} \left(k_{i,1} x_i y_1 + k_{i,1} \alpha_i \binom{y_1}{2} x_2 + k_{i,2} x_i y_2 + k_{i,2} \alpha_i y_1 y_2 x_2 + k_{i,2} \binom{x_2}{2} \alpha_i y_1 \right) . \qquad \square$$

Once again, we look for a 1-cochain $\lambda : N_2 \to \mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^1)$ whose coborder equals the cocycle f. As examples have shown, it is natural to propose the following cochain:

$$\lambda : N_2 \to \mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^1) , \qquad x \mapsto (\mathrm{Lin}(\lambda), \mathrm{Tr}(\lambda)) ,$$

given by

$$\mathrm{Lin}(\lambda)(x) = \Big(-k_{3,1} x_1 - k_{3,2} \frac{x_2}{2}, \dots, -k_{n,1} x_1 - k_{n,2} \frac{x_2}{2}, 0 , - k_{2,1} x_1 - \sum_{i=3}^{n} \left(k_{i,1} \alpha_i \binom{x_1}{2} - \frac{k_{i,2} x_i}{2} + \frac{k_{i,2} \alpha_i x_2 x_1}{2} \right) \Big)$$

and

$$\mathrm{Tr}(\lambda)(x) = - \sum_{i=3}^{n} k_{i,2} \frac{x_i (x_2 - 1)}{2} . \tag{26}$$

The verification of $\partial \lambda = f$ is straightforward and can be left to the reader.

6.3. *How to combine Case 1 and Case 2*

When given a presentation of type (11) it should be noted here that one obtains presentations described in Case 1 and Case 2 by putting $k_{2,1} = 0$ and $k_{i,1} = k_{i,2} = 0$ (for $i > 2$) (Case 1) and $k_{i,j} = 0$ (for $i > j \geq 3$) (case 2).

The cocycle corresponding to the combined (general) case (11) is easily seen to be the sum of the cocycles according to the presentations of type (13) and type (14) derived from it. Constructing an affine representation for the group N_3 is then done by splitting it up in the two cases described above and following the procedure described there.

6.4. Example

To end this section we present an example showing how to operate in practice. Let us take a group

$$\begin{aligned} N_3\colon\quad \langle a_1, a_2, a_3, a_4, a_5, a_6 \mid [a_2, a_1] &= a_3 a_5^2 a_6^{-1} , \\ [a_3, a_1] &= a_6^{-2}, [a_4, a_2] = a_6 , \\ [a_5, a_4] &= a_6, [a_4, a_3] = a_6^2 \rangle . \end{aligned}$$

N_3 fits in a short exact sequence $1 \to \mathbb{Z} \to N_3 \to N_2 \to 1$, where N_2 is given by

$$N_2\colon\quad \langle a_1, a_2, a_3, a_4, a_5 \mid [a_2, a_1] = a_3 a_5^2 \rangle .$$

Note that the consistency conditions as described in (6.2) are fulfilled. Here we have three conditions to check:

$$\begin{aligned} &\alpha_4 k_{4,3} + \alpha_5 k_{5,3} = 0 + 0 = 0 , \\ &-\alpha_3 k_{4,3} + \alpha_5 k_{5,4} = -2 + 2 = 0 , \\ &-\alpha_3 k_{5,3} - \alpha_4 k_{5,4} = 0 - 0 = 0 . \end{aligned}$$

Now we point out a faithful affine representation for N_3.

Part 1: The group of type (13) corresponding to N_3 is given by

$$\langle a_1, a_2, a_3, a_4, a_5, a_6 \mid [a_2, a_1] = a_3 a_5^2, [a_5, a_4] = a_6, [a_4, a_3] = a_6^2 \rangle .$$

This means: $k_{5,4} = 1$, $k_{4,3} = 2$ and all other $k_{i,j} = 0$.

We write down the matrix

$$(A_{i,j})_{3 \le i \le 5, 3 \le j \le 5} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

as described in (20), (21) and (22).

This will determine the linear part for the first 1-cochain (γ) we need. In the translational part of this 1-cochain we use T as described in (23) above: $T = 0$.

So, we can write down γ as $\gamma(x) = ((0, -2x_3 + x_5, 0, 0, 0), -x_5 x_4)$.

Part 2: The group of type (14) corresponding to N_3 is given by

$$\begin{aligned} \langle a_1, a_2, a_3, a_4, a_5, a_6 \mid [a_2, a_1] &= a_3 a_5^2 a_6^{-1}, \\ [a_3, a_1] &= a_6^{-2}, [a_4, a_2] = a_6 \rangle . \end{aligned}$$

Following (26), we write down the 1-cochain needed for an affine representation immediately as

$$\lambda : N_2 \to \mathrm{Aff}(\mathbb{R}^5, \mathbb{R}^1)\,,$$
$$x \mapsto \left(\left(2x_1, -\frac{x_2}{2}, 0, 0, x_1 + 2\binom{x_1}{2} + \frac{x_4}{2}\right), \frac{x_4(1-x_2)}{2}\right).$$

It is easy now to add γ and λ defined above and to obtain in this way the 1-cochain

$$x \mapsto \left(\left(2x_1, -\frac{x_2}{2} - 2x_3 + x_5, 0, 0, x_1^2 + \frac{x_4}{2}\right), -x_4x_5 + \frac{x_4(1-x_2)}{2}\right).$$

Applying (2) N_3 gets an affine 6-dimensional representation $\rho' : N_3 \to \mathrm{Aff}(\mathbb{R}^6)$ so that, if $x = a_1^{x_1}a_2^{x_2}a_3^{x_3}a_4^{x_4}a_5^{x_5}a_6^{x_6}$ then $\rho'(x)$ becomes

$$\left(\begin{pmatrix} 1 & 2x_1 & -\frac{x_2}{2} - 2x_3 + x_5 & 0 & 0 & -x_1^2 + \frac{x_4}{2} \\ 0 & 1 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2x_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -x_2x_1^2 + 2x_1x_3 - \frac{x_4x_2}{2} - 2x_3x_4 + \frac{x_4}{2} + x_6 \\ -x_2x_1 + x_3 \\ x_4 \\ -2x_2x_1 + x_5 \\ x_1 \\ x_2 \end{pmatrix}\right).$$

7. Nilpotency class ≥3 examples

In this section we describe a class of nilpotent groups of arbitrary large class where a detailed investigation of the iteration principal of Section (3) can be carried out. We will write down only the main result, without proof.

Choose a sequence of integers $\alpha_3, \alpha_4, \ldots$, and define the groups

$$N_n\colon\quad \langle a_1, \ldots, a_n \mid [a_2, a_1] = a_3^{\alpha_3}, [a_3, a_1] = a_4^{\alpha_4}, \ldots, [a_{n-1}, a_1] = a_n^{\alpha_n}\rangle\,.$$

Definition 7.1. (1) For each $m \geq 1$ we define $P_m(x) = \binom{-x}{m}$, which is to be considered as an integer-valued polynomial with rational coefficients.

(2) For $n \leq m$, we write $\alpha_{m,n} = \alpha_m.\alpha_{m-1}\ldots\alpha_n$. If $n = m$, we use $\alpha_{m,m} = \alpha_m$.

(3) For $n \geq 3$ we define

$$D_n = \left[-\alpha_n x_1, -\alpha_{n,n-1}\binom{x_1}{2}, \ldots, -\alpha_{n,4}\binom{x_1}{n-3}, -\alpha_{n,3}\binom{x_1}{n-2}, 0\right].$$

(4) Let $Q_1 = x_2$ and define inductively for $m > 1$ a polynomial Q_m by

$$Q_m = D_{m+1}\begin{pmatrix} Q_{m-1} \\ Q_{m-2} \\ \vdots \\ Q_2 \\ Q_1 \\ x_1 \end{pmatrix} + x_{m+1}\,.$$

With these notations and definitions one can prove that the groups N_n have a canonical representation $\rho : N_n \to \mathrm{Aff}(\mathbb{R}^n)$, $x = a_1^{x_1}a_2^{x_2}\ldots a_n^{x_n} \mapsto \rho(x)$ given by

$$\rho(x) = \left(\begin{bmatrix} 1 & \alpha_n P_1(x_1) & \alpha_{n,n-1}P_2(x_1) & \ldots & \alpha_{n,4}P_{n-3}(x_1) & \alpha_{n,3}P_{n-2}(x_1) & 0 \\ 0 & 1 & \alpha_{n-1}P_1(x_1) & \ldots & \alpha_{n-1,4}P_{n-4}(x_1) & \alpha_{n-1,3}P_{n-3}(x_1) & 0 \\ 0 & 0 & 1 & & & & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & \alpha_4 P_1(x_1) & \alpha_{4,3}P_2(x_1) & 0 \\ 0 & 0 & 0 & \ldots & 1 & \alpha_3 P_1(x_1) & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} Q_{n-1} \\ Q_{n-2} \\ Q_{n-3} \\ \vdots \\ Q_3 \\ Q_2 \\ Q_1 \\ x_1 \end{bmatrix}\right).$$

Example 7.2. Consider the following group:

$$N\colon \quad \langle \mathrm{a}_1, \ldots, \mathrm{a}_6 \mid [a_2, a_1] = a_3^2, [a_3, a_1] = a_4^{-1}, \\ [a_4, a_1] = a_5^3, [a_5, a_1] = a_6^{-2} \rangle\,.$$

We calculate

$$P_1(x_1) = -x_1\,,$$
$$P_2(x_1) = x_1(x_1 + 1)/2\,,$$
$$P_3(x_1) = -x_1(x_1^2 + 3x_1 + 2)/6\,,$$
$$P_4(x_1) = x_1(x_1^3 + 6x_1^2 + 11x_1 + 6)/24$$

and so we get an embedding

$$\rho : N \to \mathrm{Aff}(\mathbb{R}^6)\,,$$
$$n = a_1^{x_1}\ldots a_6^{x_6} \mapsto \rho(n) =$$
$$\left(\begin{bmatrix} 1 & 2x_1 & -3x_1(x_1+1) & -x_1(x_1^2+3x_1+2) & \frac{1}{2}x_1(x_1^3+6x_1^2+11x_1+6) & 0 \\ 0 & 1 & -3x_1 & \frac{-3}{2}x_1(x_1+1) & x_1(x_1^2+3x_1+2) & 0 \\ 0 & 0 & 1 & x_1 & -x_1(x_1+1) & 0 \\ 0 & 0 & 0 & 1 & -2x_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} Q_5 \\ Q_4 \\ Q_3 \\ Q_2 \\ Q_1 \\ x_1 \end{bmatrix}\right)$$

where

$$
\begin{aligned}
Q_1 &= x_2 , \\
Q_2 &= -2x_1x_2 + x_3 , \\
Q_3 &= -x_1(x_1 + 1)x_2 + x_1x_3 + x_4 , \\
Q_4 &= x_1(x_1^2 + 3x_1 + 2)x_2 - \tfrac{3}{2}x_1(x_1 + 1)x_3 - 3x_1x_4 + x_5 , \\
Q_5 &= \tfrac{1}{2}x_1(x_1^3 + 6x_1^2 + 11x_1 + 6)x_2 - x_1(x_1^2 + 3x_1 + 2)x_3 \\
&\quad - 3x_1(x_1 + 1)x_4 + 2x_1x_5 + x_6 .
\end{aligned}
$$

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